A Proof of Erdös - Faber - Lovász Conjecture

Suresh M. H., V. V. P. R. V. B. Suresh Dara*

Department of Mathematical and Computational Sciences,
National Institute of Technology Karnataka, Surathkal,
Mangalore - 575025, India.

Abstract

In 1972, Erdös - Faber - Lovász (EFL) conjectured that, if **H** is a linear hypergraph consisting of n edges of cardinality n, then it is possible to color the vertices with n colors so that no two vertices with the same color are in the same edge. In 1978, Deza, Erdös and Frankl had given an equivalent version of the same for graphs: Let $G = \bigcup_{i=1}^{n} A_i$ denote a graph with n complete graphs $A_1, A_2, \ldots, A_n$, each having exactly n vertices and have the property that every pair of complete graphs has at most one common vertex, then the chromatic number of G is n.

The clique degree $d^K(v)$ of a vertex v in G is given by $d^K(v) = |\{A_i : v \in V(A_i), 1 \leq i \leq n\}|$. In this paper we give an algorithmic proof of the conjecture using the symmetric latin squares and clique degrees of the vertices of G.

Keywords: Chromatic number, Erdös - Faber - Lovász conjecture, Latin squares

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*Corresponding author

Email addresses: smhegde@nitk.ac.in (Suresh M. H.), suresh.dara@gmail.com (V. V. P. R. V. B. Suresh Dara)

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1. Introduction

One of the famous conjectures in graph theory is Erdös - Faber - Lovász conjecture. It states that if $\mathbf{H}$ is a linear hypergraph consisting of n edges of cardinality n, then it is possible to color the vertices of $\mathbf{H}$ with n colors so that no two vertices with the same color are in the same edge [1]. Erdös, in 1975, offered 50 pounds [2, 3] and in 1981, offered 500USD [4, 5] for the proof or disproof of the conjecture. Kahn [6] showed that the chromatic number of $\mathbf{H}$ is at most $n + o(n)$. Jakson et al. [7] proved that the conjecture is true when the partial hypergraph S of $\mathbf{H}$ determined by the edges of size at least three can be Δ_S-edge-colored and satisfies $\Delta_S \leq 3$. In particular, the conjecture holds when S is unimodular and $\Delta_S \leq 3$. Viji Paul et al. [8] established the truth of the conjecture for all linear hypergraphs on n vertices with $\Delta(\mathbf{H}) \leq \sqrt{n + \sqrt{n} + 1}$. Sanhez - Arrayo [3] proved the conjecture for dense hypergraphs. Faber [9] proves that for fixed degree, there can be only finitely many counterexamples to EFL on this class (both regular and uniform) of hypergraphs. We consider the equivalent version of the conjecture for graphs given by Deza, Erdös and Frankl in 1978 [10, 3, 5, 11].

Conjecture 1.1. *Let $G = \bigcup_{i=1}^{n} A_i$ denote a graph with n complete graphs $(A_1, A_2, \ldots, A_n)$, each having exactly n vertices and have the property that every pair of complete graphs has at most one common vertex, then the chromatic number of G is n.*

Definition 1.2. *Let $G = \bigcup_{i=1}^{n} A_i$ denote a graph with n complete graphs $A_1, A_2, \ldots, A_n$, each having exactly n vertices and the property that every pair of complete graphs has at most one common vertex. The clique degree $d^K(G)$ of a vertex v in G is given by $d^K(v) = |\{A_i : v \in V(A_i), 1 \leq i \leq n\}|$. The maximum clique degree $\Delta^K(G)$ of the graph G is given by $\Delta^K(G) = max_{v \in V(G)} d^K(v)$.*

From the above definition one can observe that degree of a vertex in hypergraph is same as the clique degree of a vertex in a graph.

Definition 1.3. *Let G_1 and G_2 be two vertex disjoint graphs, and let x_1, x_2 be two vertices of G_1, G_2 respectively. Then, the graph $G(x_1x_2)$ obtained by merging the vertices x_1 and x_2 into a single vertex is called the concatenation of G_1 and G_2 at the points x_1 and x_2 (see [12]).*

Definition 1.4. *A latin square is an $n \times n$ array containing n different symbols such that each symbol appears exactly once in each row and once in each column. Moreover, a latin square of order n is an $n \times n$ matrix $M = [m_{ij}]$ with entries from an n-set $V = \{1, 2, \ldots, n\}$, where every row and every column is a permutation of V (see [13]). If the matrix M is symmetric, then the latin square is called symmetric latin square.*

2. Results

We know that a symmetric $n \times n$ matrix is determined by $\frac{n(n+1)}{2}$ scalars. Using symmetric latin squares we give an n-coloring of H_n constructed below. Then using the n-coloring of H_n we give an n-coloring of all the other graphs G satisfying the hypothesis of Conjecture 1.1.

Construction of H_n:

Let n be a positive integer and $B_1, B_2, \ldots, B_n$ be n copies of K_n. Let the vertex set $V(B_i) = \{a_{i,1}, a_{i,2}, a_{i,3}, \ldots, a_{i,n}\}$, $1 \leq i \leq n$.

Step 1. Let $H^1 = B_1$.

Step 2. Consider the vertices $a_{1,2}$ of H^1 and $a_{2,1}$ of B_2. Let $b_{1,2}$ be the vertex obtained by the concatenation of the vertices $a_{1,2}$ and $a_{2,1}$. Let the resultant graph be H^2.

Step 3. Consider the vertices $a_{1,3}$, $a_{2,3}$ of H^2 and $a_{3,1}$, $a_{3,2}$ of B_3. Let $b_{1,3}$ be the vertex obtained by the concatenation of vertices $a_{1,3}$, $a_{3,1}$ and let $b_{2,3}$ be the vertex obtained by the concatenation of vertices $a_{2,3}$, $a_{3,2}$. Let the resultant graph be H^3.

Continuing in the similar way, at the nth step we obtain the graph $H^n = H_n$ (for the sake of convenience we take H^n as H_n).

By the construction of H_n one can observe the following:

1. H_n is a connected graph and satisfying the hypothesis of Conjecture 1.1.
2. H_n has exactly n verticies of clique degree one and $\frac{n(n-1)}{2}$ vertices of clique degree 2 (each B_i has exactly $(n-1)$ vertices of clique degree 2 and one vertex of clique degree one, $1 \leq i \leq n$).
3. $H_n = \bigcup_{i=1}^{n} B_i$, where $B_i = A_i$ and B_i, B_j have exactly one common vertex for $1 \leq i < j \leq n$.
4. H_n has exactly $\frac{n(n+1)}{2}$ vertices.
5. One can observe that in a connected graph G if clique degree increases the number of vertices also increases, from this it follows that, H_n is the graph with minimum number of vertices satisfying the hypothesis of Conjecture 1.1. If all the vertices of G are of clique degree one, then G will have n^2 vertices. Thus, $\frac{n(n+1)}{2} \leq |V(G)| \leq n^2$.

Lemma 2.1. *If G is a graph satisfying the hypothesis of Conjecture 1.1, then G can be obtained from H_n for some n in $\mathbb{N}$.*

Proof: Let G be a graph satisfying the hypothesis of Conjecture 1.1. Let b_x be the new labeling to the vertices v of clique degree greater than one in G, where $x = \{i : \text{vertex } v \text{ is in } A_i\}$. Define $N_i = \{b_x : |x| = i\}$ for $i = 2, 3, \ldots, n$. Then the graph G is constructed from H_n as given below:

Step 1: For every common vertex $b_{i,j}$ in H_n which is not in N_2, split the vertex $b_{i,j}$ into two vertices $u_{i,j}, u_{j,i}$ such that vertex $u_{i,j}$ is adjacent only to the vertices of B_i and the vertex $u_{j,i}$ is adjacent only to the vertices of B_j in H_n.

Step 2: For every vertex b_x in N_i where $i = 3, 4, \ldots, n$, merge the vertices $u_{l_1,l_2}, u_{l_2,l_3}, \ldots, u_{l_{m-1},l_m}, u_{l_m,l_1}$ into a single vertex u_x in H_n where $l_i \in x$ and $l_i < l_j$ for $i < j$.

Let G' be the graph obtained in Step 2. Let $V(B_i')$, $V(A_i')$ be the set of all clique degree 1 vertices of B_i of G', A_i of G respectively, $1 \leq i \leq n$. Thus

by splitting all the common vertices of H_n which are not in N_2 and merging the vertices of H_n corresponding to the vertices in $N_i, i \geq 3$, we get the graph G'. One can observe that $|V(A'_i)| = |V(B'_i)|$, $1 \leq i \leq n$. Define a function $f : V(G) \to V(G')$ by

$$
\begin{aligned}
f(b_{i,j}) &= b_{i,j} && \text{for } b_{i,j} \in N_2 \\
f(b_{i_1,i_2,\ldots i_k}) &= u_{i_1,i_2,\ldots i_k} && \text{for } b_{ii_1,i_2,\ldots i_k} \in \cup_{i=3}^{n} N_i \\
f|_{V(A'_i)} &= g_i \qquad \text{(any 1-1 map } g_i : V(A'_i) \to V(B'_i)), && \text{for } 1 \leq i \leq n
\end{aligned}
$$

One can observe that f is an isomorphism from G to G'. ∎

From Lemma 2.1, one can observe that in G there are at most $\frac{n(n-1)}{2}$ common vertices.

Let G be the graph satisfying the hypothesis of Conjecture 1.1. Let $\hat{H}$ be the graph obtained by removing the vertices of clique degree one from graph G. i.e. $\hat{H}$ is the induced subgraph of G having all the common vertices of G.

Lemma 2.2. *The chromatic number of H_n is n.*

Proof: Let H_n be the graph defined as above. Let M (given below) be an $n \times n$ matrix in which an entry $m_{i,j} = b_{i,j}$, is a vertex of H_n, belonging to both B_i, B_j for $i \neq j$ and $m_{ii} = a_{i,i}$ is the vertex of H_n which belongs to B_i. i.e.,

$$
\text{M=}\begin{pmatrix}
a_{1,1} & b_{1,2} & b_{1,3} & \ldots & b_{1,n} \\
b_{1,2} & a_{2,2} & b_{2,3} & \ldots & b_{2,n} \\
b_{1,3} & b_{2,3} & a_{3,3} & \ldots & b_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{1,n} & b_{2,n} & b_{3,n} & \ldots & a_{n,n}
\end{pmatrix}
$$

Clearly M is a symmetric matrix. We know that, for every n in $\mathbb{N}$ there is a symmetric latin square (see [14]) of order $n \times n$. Bryant and Rodger [15] gave a necessary and sufficient condition for the existence of an $(n-1)$-edge coloring of K_n (n even), and n-edge coloring of K_n (n odd) using symmetric latin squares. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n and e_{ij} is the edge joining the vertices v_i and v_j of K_n, where $i < j$, then arrange the edges of K_n in the matrix form

$A = [a_{i,j}]$ where $a_{i,j} = e_{i,j}$, $a_{j,i} = e_{i,j}$ for $i < j$ and $a_{i,i} = 0$ for $1 \leq i \leq n$, we have $A = \begin{pmatrix} 0 & e_{1,2} & e_{1,3} & \dots & e_{1,n} \\ e_{1,2} & 0 & e_{2,3} & \dots & e_{2,n} \\ e_{1,3} & e_{2,3} & 0 & \dots & e_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{1,n} & e_{2,n} & e_{3,n} & \dots & 0 \end{pmatrix}$

and let V is a matrix given by

$V = \begin{pmatrix} v_1 & 0 & 0 & \dots & 0 \\ 0 & v_2 & 0 & \dots & 0 \\ 0 & 0 & v_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & v_n \end{pmatrix}$. Then, define a matrix A' as

$$A' = A + V = \begin{pmatrix} v_1 & e_{1,2} & e_{1,3} & \dots & e_{1,n} \\ e_{1,2} & v_2 & e_{2,3} & \dots & e_{2,n} \\ e_{1,3} & e_{2,3} & v_3 & \dots & e_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{1,n} & e_{2,n} & e_{3,n} & \dots & v_n \end{pmatrix}.$$

Let $C = [c_{i,j}]$ be a matrix where $c_{i,j}$ $(i \neq j)$, is the color of $e_{i,j}$ (i.e., $c_{i,j} = c(e_{i,j})$) and $c_{i,i}$ is the color of v_i. We call C the color matrix of A'. Then C is the symmetric latin square (see[15]). As the elements of M are the vertices of H_n, one can assign the colors to the vertices of H_n from the color matrix C, by the color $c_{i,j}$ (where $c_{i,j}$ denotes the value at the (i,j)-th entry in the color matrix C), for $i, j = 1, 2, \dots, n$ and $i \neq j$ to the vertex $b_{i,j}$ in H_n and the color $c_{i,i}$ (where $c_{i,i}$ denotes the value at the (i,i)-th entry in the color matrix C), for $i = 1, 2, \dots n$ to the vertex $a_{i,i}$ in H_n. Hence H_n is n colorable. ∎

As H_n is the graph satisfying the hypothesis of Conjecture 1.1. With using the coloring of H_n which is the graph satisfying the hypothesis of Conjecture 1.1 we extend the n-coloring of all possible graphs G satisfying the hypothesis of Conjecture 1.1.

Theorem 2.3. *If G is a graph satisfying the hypothesis of Conjecture 1.1, then*

G is n-colorable.

Proof: Let G be a graph satisfying the hypothesis of Conjecture 1.1. Let $\hat{H}$ be the induced subgraph of G consisting of the vertices of clique degree greater than one in G. For every vertex v of clique degree greater than one in G, label the vertex v by u_A where $A = \{i : v \in A_i; i = 1, 2, \ldots, n\}$. Define $X = \{b_{i,j} : A_i \cap A_j = \emptyset\}$, $X_i = \{v \in G : d^K(v) = i\}$ for $i = 1, 2, \ldots, m$.

Let $1, 2, \ldots, n$ be the n-colors and C be the color matrix(of size $n \times n$) as defined in the proof of Lemma 2.2. The following construction applied on the color matrix C, gives a modified color matrix C_M, using which we assign the colors to the graph $\hat{H}$. Then this coloring can be extended to the graph G. Construct a new color matrix C_1 by putting $c_{i,j} = 0, c_{j,i} = 0$ for every $b_{i,j}$ in X. Also, let $c_{i,i} = 0$ for each $i = 1, 2, \ldots, n$

Construction:

Let $T = \cup_{i=3}^{n} X_i$, $P = \emptyset$, $T'' = X_2$ and $P'' = \emptyset$.

Step 1: If $T = \emptyset$, let C_m be the color matrix obtained in Step 4 and go to Step 5. Otherwise, choose a vertex $u_{i_1,i_2,\ldots,i_m}$ from T, where $i_1 < i_2 < \cdots < i_m$, and then choose $\binom{m}{2}$ vertices b_{i_1,i_2}, b_{i_1,i_3}, $\ldots$, b_{i_1,i_m}, b_{i_2,i_3}, $\ldots$, b_{i_{m-1},i_m} from $V(H_n)$ corresponding to the set $\{i_1, i_2, \ldots, i_m\}$. Take $T' = \{b_{i_1,i_2}, b_{i_1,i_3}, \ldots, b_{i_1,i_m}, b_{i_2,i_3}, \ldots, b_{i_{m-1},i_m}\}$ and $P' = \emptyset$. Let $T'_1 = \{b_{i,j} : b_{i,j} \in T', c(b_{i,j})$ appears more than once in the i^{th} row or j^{th} column in $C\}$ and $T'_2 = \{b_{i,j} : b_{i,j} \in T', c(b_{i,j})$ appears exactly once in the i^{th} row and j^{th} column in $C\}$. If $T'_1 \neq \emptyset$ choose a vertex $b_{s,t}$ from T'_1, otherwise choose a vertex $b_{s,t}$ from T'_2. Then add the vertex $b_{s,t}$ to P' and remove it from T'. Go to Step 2.

Step 2: If $T'_2 \neq \emptyset$ go to Step 3. Otherwise, choose a vertex b_{i_{m-1},i_m} from T'_1. Let $A = \{c_{i,j} : c_{i,j} \neq 0; i = i_{m-1}, 1 \leq j \leq n\}$, $B = \{c_{i,j} : c_{i,j} \neq 0; j = i_m, 1 \leq i \leq n\}$. If $|A \cap B| < n$ then, construct a new color matrix C_2, replacing c_{i_{m-1},i_m}, $c_{i_m,i_{m-1}}$ by x, where $x \in \{1, 2, \ldots, n\} \backslash A \cup B$. Then add the vertex b_{i_{m-1},i_m} to T'_2 and remove it from T'_1. Go to Step 3. Otherwise

choose a color x which appears exactly once either in i_{m-1}^{th} row or in i_m^{th} column of the color matrix and construct a new color matrix C_2 replacing c_{i_{m-1},i_m}, $c_{i_m,i_{m-1}}$ by x. Then add the vertex b_{i_{m-1},i_m} to T_2' and remove it from T_1'. Go to Step 3.

Step 3: If $T' = \emptyset$, then add the vertex $u_{i_1,i_2,\ldots,i_m}$ to P and remove it from T, go to Step 1. Otherwise, if $T' \cap T_1' \neq \emptyset$ choose a vertex $b_{i,j}$ from $T' \cap T_1'$, if not choose a vertex $b_{i,j}$ from $T' \cap T_2'$. Go to Step 4.

Step 4: Let $c(b_{i,j}) = x,\ c(b_{s,t}) = y$. If $c(b_{i,j}) = c(b_{s,t})$, then add the vertex $b_{i,j}$ to P' and remove it from T'. Go to Step 3. Otherwise, let $A = \{c_{l,m} : c_{l,m} = x\}$, $B = \{c_{l,m} : c_{l,m} = y\} \setminus \{c_{l,m}, c_{m,l} : b_{l,m} \in P', l < m\}$. Construct a new color matrix C_3 by putting $c_{l,m} = y$ for every $c_{l,m}$ in A and $c_{l,m} = x$ for every $c_{l,m}$ in B. Then add the vertex $b_{i,j}$ to P' and remove it from T'. Go to Step 3.

Step 5: If $T'' = \emptyset$ consider $C_M = C_{m_1}$ stop the process. Otherwise, choose a vertex $u_{i,j}$ from T'' and go to Step 6.

Step 6: If $c_{i,j}$ appears exactly once in both i^{th} row and j^{th} column of the color matrix C_m, then add the vertex $b_{i,j}$ to P'' and remove it from T'', go to Step 5. Otherwise let $A = \{c_{i,j} : c_{i,j} \neq 0; 1 \leq j \leq n\}$, $B = \{c_{i,j} : c_{i,j} \neq 0; 1 \leq i \leq n\}$. Construct a new color matrix C_{m_1} by putting x in $c_{i,j}$, $c_{j,i}$ where $x \in \{1, 2, \ldots, n\} \setminus A \cup B$. Then add the vertex $u_{i,j}$ to P'' and remove it from T'', go to Step 5.

Thus, in step 6, we get the modified color matrix C_M. Then, color the vertex v of $\hat{H}$ by $c_{i,j}$ of C_M, whenever $v \in A_i \cap A_j$. Then, extend the coloring of $\hat{H}$ to G by assigning the remaining colors which are not used for A_i from the set of n-colors, to the vertices of clique degree one in A_i, $1 \leq i \leq n$. Thus G is n-colorable. ∎

Corollary 2.4. *[3] Consider a linear hypergraph* $\boldsymbol{H}$ *consisting of* n *edges each of size at most* n *and* $\delta(\boldsymbol{H}) \geq 2$. *If* H *is dense then* $\chi(\boldsymbol{H}) \leq n$.

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